# CLT for the zeros of Kostlan Shub Smale random polynomials

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**Abstract**

In this paper we find the asymptotic main term of the variance of the number of roots of Kostlan-Shub-Smale random polynomials and prove a central limit theorem for the number of roots as the degree goes to infinity.

**Resumé**

Dans ce papier nous trouvons le terme asymptotique dominant de la variance du nombre de racines réels des polynômes aléatoires de Kostlan-Shub-Smale et démontrons un théorème de la limite centrale pour ce nombre de racines.

Consider the Kostlan-Shub-Smale (KSS for short) ensemble of random polynomials:

$$X_d(x) := \sum_{n=0}^{d} a_n^{(d)}\, x^n; \quad x \in \mathbb{R},$$

where $d$ is the degree of the polynomial and the coefficients $(a_n^{(d)})$ are independent centered Gaussian random variables whose variances are the binomial coefficients, more precisely $\operatorname{Var}(a_n^{(d)}) = \binom{d}{n}$.

Denote by $N_d$ the number of real roots of $X_d$, that is

$$N_d := \#\{x \in \mathbb{R} : X_d(x) = 0\}.$$

It is well known that $\mathbb{E}(N_d) = \sqrt{d}$ [10, 15]. The aim of this paper is to prove the following result.

**Theorem 1.** *The variance of the number of real roots $N_d$ of KSS random polynomials verifies*

$$\lim_{d\to\infty} \frac{\operatorname{Var}(N_d)}{\sqrt{d}} = \sigma^2,$$

*with $0 < \sigma^2 < \infty$ given in Proposition 1 ($\sigma^2 \approx 0.57\ldots$). Furthermore, $N_d$ verifies the CLT*

$$\frac{N_d - \sqrt{d}}{d^{1/4}} \underset{d\to\infty}{\Rightarrow} N(0;\sigma^2).$$

The number of roots of random polynomials has been under the attention of physicists and mathematicians for a long time. The first results for particular choices of the coefficients are due to Bloch and Polya [5] in 1932. After many successive improvements and generalisations, in 1974 Maslova [12] stated the CLT for the number of zeros for polynomials with i.i.d. centered coefficients with finite variance. For related results see the review by Bharucha-Reid and Sambandham [6] or the Introduction in [2] and references therein.

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The study of Kostlan-Shub-Smale ($m \times m$ systems) of polynomials started in the early nineties by Kostlan [10], Bogomolny, Bohigas and Lobœuf [4] and Shub and Smale [15]. The mean number of roots [15, 10], some asymptotics as $m \to \infty$ for the variance [16] and for the probability of not having any zeros on intervals [14] are known. See also the review by Kostlan [11] and references therein.

We restrict our attention to the case $m = 1$. The mean number of real roots is $\sqrt{d}$. This fact shows a remarkable difference with the polynomials with i.i.d. centered coefficients which asymptotic mean number of roots is $2\log(d)/\pi$.

Our tools are the Rice formulas for the (factorial) moments of the number of roots [3, 7]; Kratz-León's version of the chaotic expansions for the number of zeros [8], Kratz-León method [9] and the Fourth Moment Theorem [13]. This method has been applied to trigonometric polynomials by Azaïs, Dalmao and León [2].

The paper is organised as follows. Section 1 contains some preliminaries and sets the problem in a more convenient way. Section 2 deals with the asymptotic behaviour of the variance of $N_d$. In Section 3 the asymptotic normality of the standardised $N_d$ is obtained. Section 4 contains the proofs of some auxiliary lemmas.

## 1 Preliminaries

By the binomial theorem, the covariance of the KSS polynomials is

$$\operatorname{cov}(X_d(x), X_d(y)) = \sum_{n=0}^{d} \binom{d}{n} x^n y^n = (1+xy)^d.$$

This fact suggest to homogenise the polynomials, that is, to introduce an auxiliary variable $x_0$ and to consider the polynomials

$$X_d^0(x_0, x) = \sum_{n=0}^{d} a_n x^n x_0^{d-n};$$

with $x_0,\, x \in \mathbb{R}$. The polynomial $X_d^0$ is homogeneous, that is $X_d^0(\lambda x_0, \lambda x) = \lambda^d X_d^0(x_0, x)$ for any $\lambda$; $x_0,\, x \in \mathbb{R}$. Therefore, we can think of $X_d^0$ as acting on the unit circumference $S^1$. The covariance in this case gives

$$\operatorname{cov}(X_d^0(x_0, x), X_d^0(y_0, y)) = \sum_{n=0}^{d} \binom{d}{n} (xy)^n (x_0 y_0)^{d-n} = (xy + x_0 y_0)^d = \langle (x_0, x), (y_0, y) \rangle^d,$$

where $\langle , \rangle$ stands for the usual inner product in $\mathbb{R}^2$.

Furthermore, denoting by $N_Y(I)$ the number of zeros of the process $Y$ on the set $I$; it is easy to see that

$$2N_d(\mathbb{R}) = N_{X_d^0}(S^1).$$

Now, it is convenient to write $X_d^0$ again as a polynomial in one variable by identifying $(x_0, x) \in S^1$ with the pair $(\sin(t), \cos(t))$ for some real $t$:

$$Y_d(t) := \sum_{n=0}^{d} a_n \cos^n(t) \sin^{d-n}(t),$$

with real $t$. It is easy to see that

$$N_d(\mathbb{R}) = N_{Y_d}([0, \pi]) \text{ almost surely.}$$

In fact, $x$ is a real root of $X_d$ if and only if the radial projections of $(1, x)$ onto $S^1$, once identified with a point $(\sin(t), \cos(t))$, are roots of $Y_d$; one of these projections correspond to $t \in [0, \pi]$.

Direct computations show that $Y_d$ is a centered stationary Gaussian process and that, for $s, t \in [0, \pi]$, its covariance function is given by

$$\operatorname{cov}(Y_d(s), Y_d(t)) = \langle (\cos(s), \sin(s)), (\cos(t), \sin(t)) \rangle^d = \cos^d(t-s).$$

Furthermore,

$$\operatorname{cov}(Y_d(s), Y_d'(t)) = -d \cos^{d-1}(t-s) \sin(t-s),$$

and

$$\operatorname{cov}(Y_d'(s), Y_d'(t)) = -d(d-1) \cos^{d-2}(t-s) \sin^2(t-s) + d \cos^d(t-s).$$

In particular, $\operatorname{Var}(Y_d(t)) = 1$ and $\operatorname{Var}(Y_d'(t)) = d$ for all $t$.

**Expectation of the number of roots of $Y_d$:**
Since the process $Y_d$ is stationary and for each fixed $t$ the random variables $Y_d(t)$ and $Y_d'(t)$ are independent centered Gaussian with variances 1 and $d$ respectively, using Rice formula [3], we have

$$\mathbb{E}\left(N_{Y_d}[0, \pi]\right) = \pi \cdot \mathbb{E}|Y_d'(0)| \cdot p_{Y_d(0)}(0) = \pi \cdot \sqrt{\frac{2}{\pi}} \sqrt{d} \cdot \frac{1}{\sqrt{2\pi}} = \sqrt{d}.$$

**Time scale and covariance limit:**
The next step is to scale the time in order to get a limit behaviour for the covariances. It is convenient to use the unit speed parametrisation, so we define

$$Z_d(t) := Y_d\left(\frac{t}{\sqrt{d}}\right).$$

The number of real roots of $X_d$ coincides almost surely with that of $Z_d$ in $[0, \sqrt{d}\pi]$, that is

$$N_d(\mathbb{R}) = N_{Z_d}([0, \sqrt{d}\pi]) \text{ almost surely.}$$

From now on, we restrict the process $Z_d$ to the interval $[0, \sqrt{d}\pi]$. Since $Y_d$ is stationary, so is $Z_d$. Let us denote by $r_d : [-\sqrt{d}\pi, \sqrt{d}\pi] \to \mathbb{R}$ the covariance function of $Z_d$, that is, $r_d(t) = \operatorname{cov}\left(Z_d(0), Z_d(t)\right)$. It follows that

$$\begin{aligned}
r_d(t) &= \cos^d\left(\frac{t}{\sqrt{d}}\right), \\
r_d'(t) &= -\sqrt{d} \cos^{d-1}\left(\frac{t}{\sqrt{d}}\right) \sin\left(\frac{t}{\sqrt{d}}\right), \\
r_d''(t) &= (d-1) \cos^{d-2}\left(\frac{t}{\sqrt{d}}\right) \sin^2\left(\frac{t}{\sqrt{d}}\right) - \cos^d\left(\frac{t}{\sqrt{d}}\right).
\end{aligned} \tag{1}$$

**Remark 1.** *Note that $r_d$ is an even function and for $t \in [0, \sqrt{d}\pi/2]$ we have $r_d(\sqrt{d}\pi - t) = (-1)^d r_d(t)$. This will imply that it suffices to deal with $r_d$ restricted to the interval $[0, \sqrt{d}\pi/2]$, as we shall see in Lemma 1.*

## 2 Asymptotic variance

We need to prepare some preliminaries.

**Lemma 1.** *We have*

$$\mathbb{E}(N_d(N_d - 1)) = \frac{2\sqrt{d}}{\pi} \int_0^{\sqrt{d}\pi/2} g_d(t) \left[\sqrt{1 - \rho_d^2(t)} + \rho_d(t) \arctan\left(\frac{\rho_d(t)}{\sqrt{1 - \rho_d^2(t)}}\right)\right] dt + 1. \tag{2}$$

*Here $g_d(t) = 2\pi p_d(t) v_d(t)$, being $p_d(t) = p_{Z_d(0),Z_d(t)}(0,0)$ the joint density of $Z_d(0), Z_d(t)$ evaluated at $(0,0)$; $v_d(t)$ the conditional variance of $Z_d'(0)$ (and of $Z_d'(t)$) conditioned to $Z_d(0) = Z_d(t) = 0$ and $\rho_d(t)$ the conditional correlation between the derivatives $Z_d'(0)$ and of $Z_d'(t)$ conditioned to $Z_d(0) = Z_d(t) = 0$. We have*

$$
\begin{aligned}
v_d(t) &= 1 - \frac{d\cos^{2d-2}\left(\frac{t}{\sqrt{d}}\right)\sin^2\left(\frac{t}{\sqrt{d}}\right)}{1-\cos^{2d}\left(\frac{t}{\sqrt{d}}\right)},\\
p_d(t) &= \frac{1}{2\pi\sqrt{1-\cos^{2d}\left(\frac{t}{\sqrt{d}}\right)}},\\
\rho_d(t) &= \cos^{d-2}\left(\frac{t}{\sqrt{d}}\right)\frac{1-d\sin^2\left(\frac{t}{\sqrt{d}}\right)-\cos^{2d}\left(\frac{t}{\sqrt{d}}\right)}{1-\cos^{2d}\left(\frac{t}{\sqrt{d}}\right)-d\cos^{2d-2}\left(\frac{t}{\sqrt{d}}\right)\sin^2\left(\frac{t}{\sqrt{d}}\right)}.
\end{aligned}
$$

Now, we pass to the asymptotic variance of $N_d = N_{Z_d}([0,\sqrt{d}\pi])$. We need the following asymptotics and bounds.

**Lemma 2.** *For each fixed $t \in \mathbb{R}$, we have*

$$\cos^d\left(\frac{t}{\sqrt{d}}\right) \underset{d\to\infty}{\longrightarrow} e^{-t^2/2}.$$

*Besides, these convergences are uniform in compacts. Furthermore, for $0 < a < 1$ we have the following upper bounds*

$$\cos^d\left(\frac{t}{\sqrt{d}}\right) \le \begin{cases} e^{-\alpha t^2/2}; & \text{if } 0 \le t < a\sqrt{d},\\ \cos^d(a); & \text{if } a\sqrt{d} \le t \le \pi\sqrt{d}/2. \end{cases}$$

*with $\alpha = 1 - a^2/3 \in (2/3, 1)$.*

**Remark 2.** *It is worth to say that this limit covariance defines a centered stationary Gaussian process $X$ on $[0,\infty)$. The asymptotic behaviour of the number of real roots of $X_d$ is intimately related to the asymptotic behaviour of the number of roots of $X$ in increasing intervals. Similar situations occur in [1] (where this fact is indeed used explicitly to obtain the CLT) and [2]. The fact that the limit process $X$ has Gaussian covariance function, and thus Gaussian spectral density, is remarkable.*

*Nevertheless, we do not need this fact in the sequel.*

**Lemma 3.** *For fixed $t$,*

$$
\begin{aligned}
g_d(t) = 2\pi\cdot v_d(t)\cdot p_d(t) &\underset{d\to\infty}{\longrightarrow} \frac{1-(1+t^2)e^{-t^2}}{(1-e^{-t^2})^{3/2}} =: g(t);\\
\rho_d(t) &\underset{d\to\infty}{\longrightarrow} e^{-t^2/2}\frac{1-t^2-e^{-t^2}}{1-e^{-t^2}-t^2e^{-t^2}} =: \rho(t).
\end{aligned}
$$

*Besides, $0 \le g(t) < 1$, $|\rho(t)| \le 1$ and $g(t) \to_{t\to 0} 0$. Furthermore, there exists an integrable upper bound for the r.h.s. of Equation 2.*

**Proposition 1** (limit variance for $N_d$)**.** *As $d \to \infty$ we have*

$$\frac{\operatorname{Var}(N_d)}{\sqrt{d}} \underset{d\to\infty}{\to} \sigma^2 := \frac{2}{\pi}\int_0^\infty \left(g(t)\left[\sqrt{1-\rho^2(t)} + \rho(t)\arctan\left(\frac{\rho(t)}{\sqrt{1-\rho^2(t)}}\right)\right] - 1\right)dt + 1,$$

*where $g$ and $\rho$ are defined in Lemma 3. Furthermore, $\sigma^2 < \infty$.*

**Remark 3.** *Using Mehler formula we can write also*

$$\sigma^2 = \int_0^\infty \sum_{\ell=0}^\infty \frac{a_{2\ell}^2}{(2\ell)!}\rho^{2\ell}(t)(g(t)-\delta_{0\ell})dt,$$

*being* $a_{2\ell} = 2(-1)^{\ell+1}/(\sqrt{2\pi}2^\ell \ell!(2\ell-1))$ *and* $\delta$ *Kronecker's delta.*

*Proof.* Recall that

$$\mathrm{Var}(N_d) = \mathbb{E}(N_d(N_d-1)) - (\mathbb{E}(N))^2 + \mathbb{E}(N).$$

From Lemma 1 the normalized second factorial moment is

$$\frac{\mathbb{E}(N_d(N_d-1))}{\sqrt{d}} = \frac{2}{\pi}\int_0^{\sqrt{d}\pi/2} g_d(t)\sqrt{1-\rho_d^2(t)}dt + \frac{2}{\pi}\int_0^{\sqrt{d}\pi/2} g_d(t)\rho_d(t)\arctan\left(\frac{\rho_d(t)}{\sqrt{1-\rho_d^2(t)}}\right)dt. \quad (3)$$

Let us look at the second integral in the r.h.s. of Equation (3). Let $a\sqrt{d} \le t \le \sqrt{d}\pi/2$. We go back in our scaling: $s \mapsto t/\sqrt{d}$; so $s \in [a, \pi/2]$. By the proof of Lemma 3 we have $|\rho_d(t)| \le \cos^{d-2}(a)$ (see Equation (8) below for the details). Also by Lemma 3 $g_d(t)$ is bounded by constant. Hence,

$$\frac{2}{\pi}\int_{a\sqrt{\pi}}^{\sqrt{d}\pi/2} g_d(t)|\rho_d(t)|\arctan\left(\frac{\rho_d(t)}{\sqrt{1-\rho_d^2(t)}}\right)dt \le \int_{a\sqrt{d}}^{\sqrt{d}\pi/2}\cos^{d-2}(a)dt \to_{d\to\infty} 0.$$

Let $0 \le t < a\sqrt{d}$. Lemma 3 gives the point-wise limit and the domination in order to obtain

$$\frac{2}{\pi}\int_0^{a\sqrt{\pi}} g_d(t)\rho_d(t)\arctan\left(\frac{\rho_d(t)}{\sqrt{1-\rho_d^2(t)}}\right)dt \to_d \frac{2}{\pi}\int_0^\infty g(t)\rho(t)\arctan\left(\frac{\rho(t)}{\sqrt{1-\rho^2(t)}}\right)dt$$

In particular, this integral is finite.

The first integral in the r.h.s. of Equation (3) cancel at infinity with $(\mathbb{E}(N_d))^2$. In fact, we can write

$$(\mathbb{E}N_d)^2 = d = \frac{2}{\pi}\sqrt{d}\int_0^{\sqrt{d}\pi/2} dt.$$

Hence, the first integral in the r.h.s. of Equation (3) minus $(\mathbb{E}(N_d))^2$ gives

$$\frac{2}{\pi}\int_0^{\pi\sqrt{d}/2}\left[g_d(t)\sqrt{1-\rho_d^2(t)}-1\right]dt = \frac{2}{\pi}\int_0^{\pi\sqrt{d}/2}\left[\frac{v_d(t)\sqrt{1-\rho_d^2(t)}}{\sqrt{1-r_d^2(t)}}-1\right]dt. \quad (4)$$

From Lemma 3 it follows that the integrand of the l.h.s of Equation (4) tends to $g(s)-1$.

In order to obtain a domination, by standard manipulation, it follows that the important point is to bound the difference $v_d^2(t)(1-\rho_d^2(t)) - 1 + r_d^2(t) = (v_d^2(t) - 1 + r_d^2(t)) - (v_d^2(t)\rho_d^2(t))$ in the numerator. The second term is easily bounded. For the first one, we have

$$v_d^2(t) - 1 + r_d^2(t) = \left[1 - \frac{d\cos^{2d-2}\left(\frac{t}{\sqrt{d}}\right)\sin^2\left(\frac{t}{\sqrt{d}}\right)}{1-\cos^{2d}\left(\frac{t}{\sqrt{d}}\right)}\right]^2 - 1 + \cos^{2d}\left(\frac{t}{\sqrt{d}}\right)$$

After expanding the squares and cancelling the ones, we divide again into the cases $0 \le t < a\sqrt{d}$ and $a\sqrt{d} \le t \le \sqrt{d}\pi/2$. In the latter, the bound $\cos(t/\sqrt{d}) < \cos(a)$ suffices to obtain that the integral tends to 0. In the former, an uniform integrable upper bound for this difference follows easily by the triangle inequality and using the bounds $d^j\sin^{2j}(t/\sqrt{d}) \le t^{2j}$; $\cos(t/\sqrt{d}) < e^{-\alpha t^2/2}$ and $\cos^{-1}(t/\sqrt{d}) < \cos^{-1}(a)$.

The denominator is handled using the bound in Lemma 2. This gives the domination, so we can pass to the limit inside the integral. Hence, the first integral in the r.h.s. of Equation (3) minus $(\mathbb{E}(N_d))^2$ tend, as $d \to \infty$, to

$$\frac{2}{\pi}\int_0^\infty (g(t)-1)dt.$$

In particular, this integral is finite, thus, so is $\sigma^2$.

Finally, let us say that the convergence at 0 of the integral in Equation (4) follows from the proof of Lemma 3. The result follows. $\square$

## 3 CLT

**Proposition 2.** *The normalised number of zeros of KSS polynomials converge in distribution towards a centered Gaussian random variable with variance $\sigma^2$.*

The idea of the proof is the following: to write the normalised number of zeros as a chaotic series [8] and then to use the Fourth Moment Theorem [13] combined with Kratz-León method [9] in order to obtain the asymptotic normality.

More precisely, we take the Itô-Wiener expansion of the normalised number of zeros [8]. Then, by Kratz-León method [9], the finiteness of the variance of $N_d$ allows to truncate the expansion and to derive its asymptotic normality from that of the sum of the first, say $Q$, terms. Finally, the Fourth Moment Theorem [13] gives a criterion to prove the asymptotic normality of the finite partial sums of the expansion.

*Proof.* We apply Kratz-León expansion [8] to the processes $Z_d$ on the interval $[0, \sqrt{d}\pi]$. Hence

$$\frac{N_d - \mathbb{E}(N_d)}{d^{1/4}} = \sum_{q=2}^{\infty} I_{q,d},$$

where

$$I_{q,d} = \frac{1}{d^{1/4}}\int_0^{\sqrt{d}\pi} f_q(Z_d(t), Z_d'(t))dt, \quad f_q(x,y) = \sum_{\ell=0}^{[q/2]} b_{q-2\ell}a_{2\ell}H_{q-2\ell}(x)H_{2\ell}(y), \tag{5}$$

with $a_{2\ell} = 2(-1)^{\ell+1}/(\sqrt{2\pi}2^\ell \ell!(2\ell-1))$, $b_k = \frac{1}{k!}\varphi(0)H_k(0)$. Note that we can delete the term corresponding to $q=1$ since $H_1(0)=0$; this is why we restrict our attention to zeros.

We can express $I_{q,d}$ as multiple stochastic integrals w.r.t $B$. In the first place, using a standard B.m. $B$ we can write $Z_d(t) = \int_{\mathbb{R}} h_d(t,\lambda)dB(\lambda)$ with

$$h_d(t,\lambda) = \sum_{n=0}^{d}\sqrt{\binom{d}{n}}\cos^n\left(\frac{t}{\sqrt{d}}\right)\sin^{d-n}\left(\frac{t}{\sqrt{d}}\right)\mathbf{1}_{[n,n+1]}(\lambda). \tag{6}$$

Then, from Equation (6), using the properties of the chaos and the stochastic Fubini theorem, see [2, Remark 2], we have $I_q = I_q^B(g_q(\boldsymbol{\lambda}_q))$ with

$$g_q(\boldsymbol{\lambda}_q) = \frac{1}{d^{1/4}}\int_0^{\sqrt{d}\pi}\sum_{j=0}^{\lfloor q/2\rfloor} b_{q-2j}a_{2j}(h_d^{\otimes q-2j}(s,\boldsymbol{\lambda}_{q-2j}) \otimes h_d'^{\otimes 2j}(s,\boldsymbol{\lambda}_{2j}))ds;$$

where $\boldsymbol{\lambda}_k \in \mathbb{R}^k$ and $\otimes$ stands for tensorial product.

Now, to get the asymptotic normality of the standardised zeros, by Kratz-León method and the Fourth Moment Theorem, it suffices to prove that the contractions $g_q \otimes_k g_q(\boldsymbol{\lambda}_{2q-2k})$ tend to 0 in $L^2$ as $d \to \infty$ for $q \geq 2$ and $k = 1, \cdots, q-1$ and $\lambda_{2q-2k} \in \mathbb{R}^{2q-2k}$.

Let $\boldsymbol{z}_k = (z_1, \ldots, z_k)$ and $\boldsymbol{\lambda}_{2q-2k} = \boldsymbol{\lambda}_{q-k} \otimes \boldsymbol{\lambda}'_{q-k}$. The contractions are defined [13] as

$$g_q \otimes_k g_q(\boldsymbol{\lambda}_{2q-2k}) = \int_{\mathbb{R}^k} g_q(\boldsymbol{z}_k, \boldsymbol{\lambda}_{q-k}) g_q(\boldsymbol{z}_k, \boldsymbol{\lambda}'_{q-k}) d\boldsymbol{z}_k.$$

Actually, by the properties of stochastic integrals, we have $I_q^B(g_q(\boldsymbol{\lambda}_q)) = I_q^B(Sym(g_q(\boldsymbol{\lambda}_q)))$ being

$$Sym(g_q)(\boldsymbol{\lambda}_q) = \frac{1}{q!} \sum_{\sigma \in S_q} g_q(\boldsymbol{\lambda}_\sigma),$$

being $S_q$ the group of permutations of $\{1, \ldots, q\}$ and $\boldsymbol{\lambda}_\sigma = (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(q)})$. So we compute contractions for $Sym(g_q)(\boldsymbol{\lambda}_q)$ instead of $g_q(\boldsymbol{\lambda}_q)$.

Writing down the norm of the contractions is quite tedious, but the basic fact is that the isometric property of stochastic integrals implies that $h_d^{\otimes p}(s) \otimes_k h_d^{\otimes p}(t) = r_d^k(t-s) h_d^{\otimes p-k}(s) \otimes h_d^{\otimes p-k}(t)$. Similarly, when the identified variable in the contraction involves the derivatives of $h$ the result involves the derivatives of $r_d$. Taking this into account, it follows that

$$\|Sym(g_q) \otimes_k Sym(g_q)(\boldsymbol{\lambda}_{2q-2k})\|_2^2 = \frac{1}{d} \iiiint_{[0,\sqrt{d}\pi]^4} \sum_{0 \le \boldsymbol{j} \le [q/2]} c_{\boldsymbol{j}} \frac{1}{q!} \sum_{\sigma \in S_q}$$
$$\prod_{i=0}^{2} (r_d^{(i)}(t-s))^{\alpha_i} (r_d^{(i)}(t'-s'))^{\beta_i} (r_d^{(i)}(s-s'))^{\gamma_i} (r_d^{(i)}(t-t'))^{\delta_i} \, ds \, dt \, ds' \, dt';$$

where $\boldsymbol{j} = (j_1, j_2, j_3, j_4)$, vector inequalities are understood component-wise; $c_{\boldsymbol{j}} = \prod_{i=1}^4 a_{2j_i} b_{q-2j_i}$; $\alpha_i = \alpha_i(\sigma, \boldsymbol{j})$, $\beta_i = \beta_i(\sigma, \boldsymbol{j})$, $\gamma_i = \gamma_i(\sigma, \boldsymbol{j})$ and $\delta_i = \delta_i(\sigma, \boldsymbol{j})$; $\sum_{i=1}^4 \alpha_i = \sum_{i=1}^4 \beta_i = k$ and $\sum_{i=1}^4 \gamma_i = \sum_{i=1}^4 \delta_i = q-k$. Actually, there are some constrains for $\alpha, \beta, \gamma, \delta$ with respect to $\boldsymbol{j}$, (namely $\alpha_1 \le (q-2j_1) \wedge (q-2j_2)$, $\alpha_2 \le (q-2j_1) \wedge 2j_2 + (q-2j_2) \wedge 2j_1$, etc), but they are irrelevant for our purposes.

We bound the covariances by their absolute value. Since $\mathrm{Var}(Z_d(t)) = \mathrm{Var}(Z'_d(t)) = 1$, by Cauchy-Schwarz, each factor $|r_d^{(i)}(\cdot)| \le 1$. Furthermore, since $k \ge 1$ and $q-k \ge 1$, we can bound from above the product of each group of factors (i.e.: with the same argument) by one of them.

Hence, for some $i_1, i_2, i_3, i_4 \in \{0, 1, 2\}$ we have

$$\|Sym(g_q) \otimes_k Sym(g_q)(\boldsymbol{\lambda}_d)\|_2^2 \le$$
$$\frac{C}{d} \iiiint_{[0,\sqrt{d}\pi]^4} |r_d^{(i_1)}(t-s) \, r_d^{(i_2)}(t'-s') \, r_d^{(i_3)}(s-s') \, r_d^{(i_4)}(t-t')| \, ds \, dt \, ds' \, dt',$$

where $C$ is a meaningless constant. Now, we make the change of variables: $(x, y, u, t') \mapsto (t-s, t'-s', s-s', t')$ and enlarge the domain of integration in order to have a rectangular one. Thus

$$\|Sym(g_q) \otimes_k Sym(g_q)(\boldsymbol{\lambda}_d)\|_2^2 \le \frac{C}{d} \int_0^{\sqrt{d}\pi} dt' \int_{-\sqrt{d}\pi}^{\sqrt{d}\pi} |r_d^{(i_1)}(x)| dx \int_{-\sqrt{d}\pi}^{\sqrt{d}\pi} |r_d^{(i_2)}(y)| dy \int_{-\sqrt{d}\pi}^{\sqrt{d}\pi} |r_d^{(i_3)}(u)| \, du.$$

Let us look at the three inner integrals. Note that since $r_d$ is even, so is the absolute value of its derivatives, so it suffices to integrate on $[0, \sqrt{d}\pi]$. Besides, since for $t \in [0, \sqrt{d}\pi/2]$ we have $r_d(\sqrt{d}\pi - x) = (-1)^d r_d(x)$, it follows that $|r_d^{(i)}(\sqrt{d}\pi - x)| = |r_d^{(i)}(x)|, i \in \{0, 1, 2\}$. Then, we can further restrict the domain of integration to $[0, \sqrt{d}\pi/2]$. The finiteness of the integral then follows from Equation (1), by bounding the covariance by a polynomial (of degree at most 2) times $\cos^d(\cdot/\sqrt{d})$ and then using Lemma 2. Hence, the contractions tend to 0.

The result follows. □

**Corollary 2.** *The asymptotic variance $\sigma^2$ is strictly positive.*

*Proof.* From the Itô-Wiener expansion (5) it follows that $\sigma^2 = \sum_{q=2}^\infty \mathrm{Var}(I_q)$ with $\mathrm{Var}(I_q) = \lim_d \mathrm{Var}(I_{q,d})$. Thus, it suffices to prove that $\mathrm{Var}(I_2) > 0$. This is done exactly as in [3, Eq 10.42-10.43]. □

*Proof of Theorem 1.* Put together Propositions 1 and 2 and Corollary 2. The approximated value for $\sigma^2$ is obtained numerically from the formula in Proposition 1. $\square$

It worth to say that this value is confirmed by simulations.

## 4 Proofs of the lemmas

*Proof of Lemma 1.* We compute the second factorial moment via Rice formula. By Equation (10.7.5) of [7] we have

$$\mathbb{E}(N_d(N_d-1)) = \frac{2}{\pi^2}\int_0^{\sqrt{d}\pi}(\sqrt{d}\pi - t)g_d(t)\left(\sqrt{1-\rho_d^2(t)} + \rho_d(t)\arctan\left(\frac{\rho_d(t)}{\sqrt{1-\rho^2(t)}}\right)\right)dt$$

Denote

$$f(t) = g_d(t)\left(\sqrt{1-\rho_d^2(t)} + \rho_d(t)\arctan\left(\frac{\rho_d(t)}{\sqrt{1-\rho^2(t)}}\right)\right).$$

Then, $f(\sqrt{d}\pi - t) = f(t)$. This follows from the properties $\cos(t) = \cos(-t) = -\cos(\pi - t)$ and $\sin(t) = -\sin(-t) = \sin(\pi - t)$ and from the fact that we only use even powers of the cosines and sines in $g_d$ and $\rho_d^2$ and that the signs also cancel in the product $\rho\arctan(\rho/\sqrt{1-\rho^2})$.

Then, using the change of variables $x = \sqrt{d}\pi - t$ in the interval $[\sqrt{d}\pi/2, \sqrt{d}\pi]$, we have

$$\begin{aligned}
\mathbb{E}(N_d(N_d-1)) &= 2\int_0^{\sqrt{d}\pi} f(t)(\sqrt{d}\pi - t)dt \\
&= 2\int_0^{\sqrt{d}\pi/2} f(t)(\sqrt{d}\pi - t)dt + 2\int_{\sqrt{d}\pi/2}^{\sqrt{d}\pi} f(t)(\sqrt{d}\pi - t)dt \\
&= 2\int_0^{\sqrt{d}\pi/2} f(t)(\sqrt{d}\pi - t)dt + 2\int_{\sqrt{d}\pi/2}^{0} f(\sqrt{d}\pi - x)(\sqrt{d}\pi - [\sqrt{d}\pi - x])(-dx) \\
&= 2\int_0^{\sqrt{d}\pi/2} f(t)(\sqrt{d}\pi - t)dt + 2\int_0^{\sqrt{d}\pi/2} f(x)x\,dx = 2\int_0^{\sqrt{d}\pi/2} f(t)\sqrt{d}\pi\,dt
\end{aligned}$$

$\square$

*Proof of Lemma 2.* We start with the cosine.

Assume that $0 \le t < a\sqrt{d}$. Using Taylor-Lagrange expansion up to the first order for the cosine, we can write

$$\cos^d\left(\frac{t}{\sqrt{d}}\right) = \left(1+\frac{x}{d}\right)^d; \text{ with } x = -\frac{t^2}{2}\left(1 - \frac{\sin(t^*)t}{3\sqrt{d}}\right),$$

where $t^* \in [0, t/\sqrt{d}] \subset [0, a]$.

There exists $c(a)$ such that $|x/d| < c(a) < 1$. (In fact, if $t < a\sqrt{d}$, then $x/d = -a^2/2 + \sin(t^*)a^3/6$. Since the summands have different signs, it follows that $-1 < -a^2/2 \le x/d \le a^4/6 < 1$. The claim follows with $c(a) = \max\{a^2/2, a^4/6\} = a^2/2$. )

Again by Taylor-Lagrange expansion, for the logarithm this time, it follows that

$$\log\left(\cos^d\left(\frac{t}{\sqrt{d}}\right)\right) = \log\left(1+\frac{x}{d}\right)^d = x + \log''(1+x^{**})\frac{x^2}{2d};$$

with $x^{**} \in [0, x/d] \subset [0, a]$. Hence,

$$\cos^d\left(\frac{t}{\sqrt{d}}\right) = \left(1+\frac{x}{d}\right)^d = e^x e^{\log''(1+x^{**})\frac{x^2}{2d}}.$$

Note that $\log''(1+x^{**}) = -(1+x^{**})^{-2} \in (-(1-a)^{-2}, -(1+a)^{-2}]$; so $\log''(1+x^{**}) < 0$.

Now, for fixed $t$ is easy to see that when $d \to \infty$ then $|t| < a\sqrt{d}$; $x \to -t^2/2$ and $\log''(1+x^{**})x^2/2d \to 0$. Thus, $\cos^d\left(\frac{t}{\sqrt{d}}\right) \to e^{-t^2/2}$ as $d \to \infty$ as claimed.

It is also easy to check the uniformity of the convergence in compacts $[0,\beta]$ from the preceding expression. In fact, let $t \in [0,\beta]$, then

$$\begin{aligned}\left|\cos^d\left(\frac{t}{\sqrt{d}}\right) - e^{-t^2/2}\right| &= e^{-t^2/2}\left|e^{\sin(t^*)t^3/6d^{3/2}+\log''(1+x^{**}(t))x^2(t)/2d} - 1\right| \\ &\leq e^{-t^2/2}\max\{\left|e^{\sin(\beta^*)\beta^3/6d^{3/2}} - 1\right|, \left|e^{-(1-\beta^2/2d)^{-2}\beta^4/4d} - 1\right|\} \to_d 0;\end{aligned}$$

where we use the fact that the summands in the exponent have different signs.

Now, we turn to the bounds. Since $\log''(1+x^{**}) < 0$ we see that $\cos^d(t/\sqrt{d}) \leq e^x$. Furthermore, since $sin(\cdot) \leq \cdot$, we have $0 \leq \sin(t^*)t/3\sqrt{d} \leq a^2/3$. Hence,

$$\cos^d\left(\frac{t}{\sqrt{d}}\right) \leq e^x = \exp\left\{-\frac{t^2}{2}\left(1 - \frac{\sin(t^*)t}{3\sqrt{d}}\right)\right\} \leq \exp\left\{-\frac{t^2}{2}\left(1 - \frac{a^2}{3}\right)\right\} \leq e^{-\alpha t^2/2}, \tag{7}$$

as claimed.

Assume now that $a\sqrt{d} \leq t \leq \sqrt{d}\pi/2$. In this case, we have $\cos(t/\sqrt{d}) < \cos(a) < 1$, hence $\cos^d(t/\sqrt{d}) < \cos^d(a)$. The result follows. ∎

*Proof of Lemma 3.* The limits are direct consequences of Lemma 2.

Bounds: Let us look at the domination in a neighbourhood of $t = 0$. First, note that $\rho_d(t) \leq 1$ since it is a correlation; so the sum is finite.

Consider the factor

$$2\pi \cdot v_d(t) \cdot p_d(t) = \frac{1 - \cos^{2d}\left(\frac{t}{\sqrt{d}}\right) - d\cos^{2d-2}\left(\frac{t}{\sqrt{d}}\right)\sin^2\left(\frac{t}{\sqrt{d}}\right)}{\left(1 - \cos^{2d}\left(\frac{t}{\sqrt{d}}\right)\right)^{3/2}}.$$

By Taylor-Lagrange expansion, as in the proof of Lemma 2, we can write

$$1 - \cos^{2d}\left(\frac{t}{\sqrt{d}}\right) = 1 - e^u = 1 - \left(1 + u + e^{u^*}\frac{u^2}{2}\right) = -u - e^{u^*}\frac{u^2}{2};$$

where

$$\begin{aligned} u &= 2x + \log''(1+x^{**})\frac{x^2}{d}; \quad x^{**} \in [0, x/d] \subset [0,x]; \\ x &= -\frac{t^2}{2} + \frac{\sin(t^*)t^3}{6\sqrt{d}}; \quad t^* \in [0, t/\sqrt{d}]. \end{aligned}$$

First, we bound $x$; since $t$ lies on a neighbourhood of 0 (assume, for simplicity, that $t < 1$), the signature of the sinus is the same as that of $t$. Hence,

$$-\frac{t^2}{2} \leq x = -\frac{t^2}{2} + \frac{\sin(t^*)t^3}{6\sqrt{d}} \leq -\frac{t^2}{2} + \frac{t^4}{6d} \leq -\frac{t^2}{2} + \frac{t^4}{6}.$$

Now, we pass to the second term in the definition of $u$: since $x < 0$ and thus $x^{**} < 0$, we have

$$-\log''(1+x^{**}) = \frac{1}{(1+x^{**})^2} \geq 1.$$

Hence,

$$-\log''(1+x^{**})\frac{x^2}{d} \geq \frac{x^2}{d} \geq \frac{1}{d}\left(-\frac{t^2}{2}+\frac{t^4}{6}\right)^2.$$

On the other hand, $x^{**} \geq x \geq -t^2/2$, so $1+x^{**} \geq 1-t^2/2$. Hence,

$$-\log''(1+x^{**})\frac{x^2}{d} \leq \frac{1}{(1-t^2/2)^2}(t^2/2)^2 \leq \frac{t^4}{4(1-t^2/2)^2}.$$

Therefore, putting these bounds together

$$-t^2 - \frac{t^4}{4(1-t^2/2)^2} \leq u \leq -t^2 + \frac{t^4}{3} - \frac{1}{d}\left(-\frac{t^2}{2}+\frac{t^4}{6}\right)^2.$$

Hence,

$$-t^2 - \frac{t^4}{4(1-t^2/2)^2} \leq u \leq -t^2 + \frac{t^4}{3}.$$

Finally, using that $e^{u^*} \leq 1$:

$$t^2 - \frac{t^4}{3} - \frac{1}{2}\left(t^2 + \frac{t^4}{4(1-t^2/2)^2}\right)^2 \leq -u - \frac{u^2}{2} \leq 1 - \cos^{2d}\left(\frac{t}{\sqrt{d}}\right) \leq -u \leq t^2 + \frac{t^4}{4(1-t^2/2)^2}.$$

Thus

$$t^2 - \frac{t^4}{3} - \frac{1}{2}\left(t^2 + \frac{t^4}{4(1-t^2/2)^2}\right)^2 \leq 1 - \cos^{2d}\left(\frac{t}{\sqrt{d}}\right) \leq t^2 + \frac{t^4}{4(1-t^2/2)^2}.$$

Similarly, we have $\cos^{2d-2}(t/\sqrt{d}) \geq \cos^{2d}(t/\sqrt{d})$. Besides,

$$\sin^2(t/\sqrt{d}) = \left(\frac{t}{\sqrt{d}} + \frac{\sin(t^{***})t^2}{2d}\right)^2, \quad t^{***} \in [0, t/\sqrt{d}].$$

Hence, using the bound for $1-\cos^{2d}$, we have

$$\begin{aligned}
-d\cos^{2d-2}\left(\frac{t}{\sqrt{d}}\right)\sin^2\left(\frac{t}{\sqrt{d}}\right) &\leq -d\cos^{2d}\left(\frac{t}{\sqrt{d}}\right)\left(\frac{t}{\sqrt{d}} + \frac{\sin(t^{***})t^2}{2d}\right)^2 \\
&\leq d\left(t^2 - 1 + \frac{t^4}{4(1-t^2/2)^2}\right)\left(\frac{t^2}{d} + \frac{\sin(t^{***})t^3}{d^{3/2}} + \frac{\sin^2(t^{***})t^4}{4d^2}\right) \\
= -t^2 + t^3\Bigg[\left[t^2 - 1 + \frac{t^4}{4(1-t^2/2)^2}\right]&\left[\frac{\sin(t^{***})}{d^{1/2}} + \frac{\sin^2(t^{***})t}{4d}\right] + t + \frac{t^3}{4(1-t^2/2)^2}\Bigg] \\
&\leq -t^2 + t^4 + \frac{t^6}{4(1-t^2/2)^2}; \text{ if } t<1.
\end{aligned}$$

Therefore, taking $t<1$

$$\begin{aligned}
2\pi \cdot v_d(t) \cdot p_d(t) &\leq \frac{\left(t^2 + \frac{t^4}{4(1-t^2/2)^2}\right) + \left(-t^2 + t^4 + \frac{t^6}{4(1-t^2/2)^2}\right)}{\left(t^2\left[1 - \frac{t^2}{3} - \frac{1}{2}\left(t + \frac{t^3}{4(1-t^2/2)^2}\right)^2\right]\right)^{3/2}} \\
&= \frac{t^4\left[1 + \frac{1+t^2}{4(1-t^2/2)^2}\right]}{t^3\left(1 - \frac{t^2}{3} - \frac{1}{2}\left(t + \frac{t^3}{4(1-t^2/2)^2}\right)^2\right)^{3/2}} = \frac{t\left[1 + \frac{1+t^2}{4(1-t^2/2)^2}\right]}{\left(1 - \frac{t^2}{3} - \frac{1}{2}\left(t + \frac{t^3}{4(1-t^2/2)^2}\right)^2\right)^{3/2}}.
\end{aligned}$$

This gives an integrable (at 0) upper bound.

Besides, Lemma 2 shows that the convergences are uniform in compacts.

Finally, it rests to obtain an integrable upper bound for large $t$ (but recall that $t \in [0, \sqrt{d}\pi/2]$). Assume that $t \geq t_0$. For $t < a\sqrt{d}$ we can use the bound $\cos^d(t/\sqrt{d}) \leq e^{-\alpha t^2/2}$. Then, the upper bound is easy to obtain:

$$2\pi \cdot v_d(t) \cdot p_d(t) \leq \frac{1 + (1+t^2)e^{-\alpha t^2}}{(1 - e^{-\alpha t^2})^{3/2}},$$
$$|\rho_d(t)| \leq \frac{e^{-\alpha t^2/2}}{\cos^2(a)} \frac{1 + t^2 + e^{-\alpha t^2}}{1 - e^{-\alpha t^2} - t^2 e^{-\alpha t^2}}.$$

For $t > a\sqrt{d}$ is similar. Let $s = t/\sqrt{d}$. Let us start with the correlation $\rho_d(t)$. Since $\cos(s) \leq \cos(a)$, we have

$$-d\cos^{2d-2}(s)\sin^2(s) + 1 - \cos^{2d}(s) \geq -d\cos^{2d-2}(a)\sin^2(a) + 1 - \cos^{2d}(a) \to_d 1,$$

so, the demominator in $\rho_d(t)$ is positive for $d$ large enough. Thus

$$\begin{aligned}|\rho_d(t)| &= \cos^{d-2}(s)\frac{|1 - d\sin^2(s) - \cos^{2d}(s)|}{1 - \cos^{2d}(s) - d\cos^{2d-2}(s)\sin^2(s)} \\ &\leq \cos^{d-2}(s)\frac{|1 - d\sin^2(s) - \cos^{2d}(s)|}{1 - d\sin^2(s) - \cos^{2d}(s)} = \cos^{d-2}(s) \leq \cos^{d-2}(a). \quad (8)\end{aligned}$$

Similarly, since $\cos(s) < \cos(a)$ in this region, we know that the conditional variance and the density (in this region) are bounded by constants (for instance, 1 and $(2\pi\sqrt{1 - \cos^{2d}(a)})^{-1}$ respectively).

This gives the necessary domination for the cases $\ell \geq 1$ and concludes the proof of the Lemma. □

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